Cantor Spectrum for the Almost Mathieu Operator. Corollaries of localization, reducibility and duality

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Abstract

In this paper we use results on reducibility, localization and duality for the Almost Mathieu operator,

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos\left(2\pi n\omega + \phi\right) x_n$$

on $l^2(\mathbb{Z})$ and its associated eigenvalue equation to deduce that for $b \neq 0, \pm 2$ and ω Diophantine the spectrum of the operator is a Cantor subset of the real line. This solves the so-called "Ten Martini Problem" for these values of b and ω. Moreover, we prove that for $|b| \neq 0$ small enough or large enough all spectral gaps predicted by the Gap Labelling theorem are open.

1 Introduction. Main results

In this paper we study the nature of the spectrum of the Almost Mathieu operator

$$(H_{b,\phi}x)_n = x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n, \qquad n \in \mathbb{Z} \tag{1}$$

on $l^2(\mathbb{Z})$, where b is a real parameter, ω is an irrational number and $\phi \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Since for each b this is a bounded self-adjoint operator, the spectrum is a compact subset of the real line which does not depend on ϕ because of the assumption on ω. This spectrum will be denoted by σ_b. For $b = 0$, this set is the interval $[-2, 2]$. The understanding of the spectrum of (1) is related to the dynamical properties of the difference equation

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z} \tag{2}$$

for $a \in \mathbb{R}$, which is sometimes called the Harper equation. In what follows we will assume that the frequency ω is Diophantine:

Definition 1 *We say that a real number ω is Diophantine whenever there exist positive constants c and $r > 1$ such that the estimate*

$$|\sin 2\pi n\omega| > \frac{c}{|n|^r}$$

holds for all $n \neq 0$.

The nature of the spectrum of this operator has been studied intensively in the last twenty years (for a review, see Last [27]) and an open problem has been to know whether the spectrum is a Cantor set or not, which is usually referred as the "Ten Martini Problem". In this paper we derive two results on this problem. The first one is *non-perturbative*:

Corollary 2 *If ω is Diophantine, then the spectrum of the Almost Mathieu operator is a Cantor set if $b \neq 0, \pm 2$.*

Here, we prefer to call this result a corollary, rather than a theorem, because the proof requires just a combination of reducibility, point spectrum and duality developed quite recently for the Almost Mathieu operator and the related eigenvalue equation. The argument is in fact reminiscent of Ince's original argument for the classical Mathieu differential equation (see [19]). In the critical case $|b| = 2$, Y. Last proved in [26] that the spectrum of the Almost Mathieu operator is a subset of the real line with zero Lebesgue measure and that it is a Cantor set for the values of ω which have an unbounded continued fraction expansion, which is a set of full measure. This last result has been extended recently to all Diophantine frequencies by Avila & Krikorian [2].

The Cantor structure of the spectrum of the Almost Mathieu operator can be better understood if we make use of the concept of *rotation number*, which can be defined as follows. Let $(x_n)_{n\in\mathbb{Z}}$ be a non-trivial solution of (2), for some fixed a, b, ϕ. Let $S(N)$ be the number of changes of sign of such solution for $1 \leq n \leq N$, adding one if $x(N) = 0$. Then the limit

$$\lim_{N\to\infty} \frac{S(N)}{2N}$$

exists, it does not depend on the chosen solution x, nor on ϕ and it is denoted by $\mathrm{rot}(a, b)$. A more complete presentation of this object can be found in section 2. Here we only mention some properties which relate it to the spectrum of $H_{b,\phi}$:

Proposition 3 ([3, 12, 18, 23]) *The rotation number has the following properties:*

(i) The rotation number, $\mathrm{rot}(a, b)$, is a continuous function of $(a, b) \in \mathbb{R}^2$.

(ii) For a fixed b, the spectrum of (1), σ_b, is the set of $a_0 \in \mathbb{R}$, such that $a \mapsto \mathrm{rot}(a, b)$ is not locally constant at a_0.

(iii) (**Gap labelling**) *If I is an open, non-void interval in the resolvent set of (1), $\rho_b = \mathbb{R} - \sigma_b$, then there is an integer $k \in \mathbb{Z}$ such that*

$$2\mathrm{rot}(a, b) - k\omega \in \mathbb{Z}$$

for all $a \in I$. That is,

$$\mathrm{rot}(a, b) = \frac{1}{2}\{k\omega\}$$

where $\{\cdot\}$ denotes the fractional part of a real number.

From this theorem we conclude that the resolvent set is the disjoint union of countably (or finitely) many open intervals called *spectral gaps*, possibly void, and which can be uniquely labelled by an integer k called the *resonance*. If the closure of a spectral gap degenerates to a point we will say that it is a *collapsed gap* and otherwise that it is a *non-collapsed gap*.

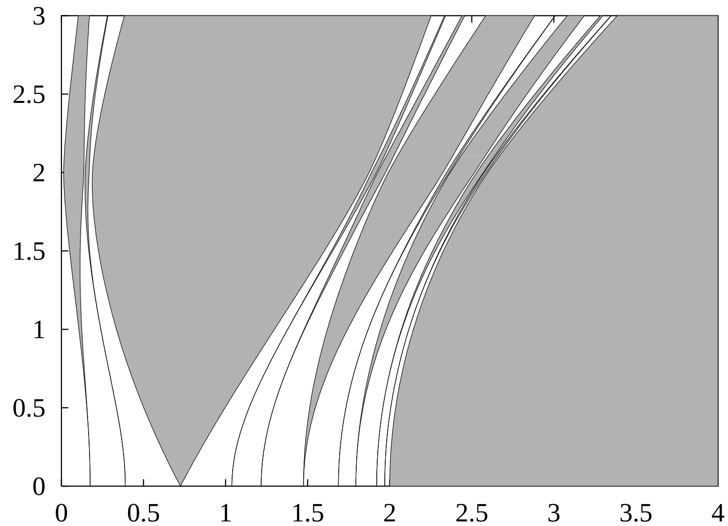

Figure 1: Numerical computation of the ten biggest spectral gaps for the Almost Mathieu operator with different values of b and $\omega = (\sqrt{5}-1)/2$. They correspond to the first $|k|$ such that $\{k\omega\}/2$ belongs to $[1/4, 1/2]$. The coupling parameter b is in the vertical direction whereas the spectral one, a, is in the horizontal one. Note that for $b = 0$, all gaps except the upper one are collapsed.

See Figure 1 for a numerical computation of the biggest gaps in the spectrum of the Almost Mathieu operator for several values b.

In particular if, for a fixed b, all the spectral gaps are open and the frequency ω is irrational, then the spectrum σ_b is a Cantor set. The question of the non-collapsing of all spectral gaps is sometimes called the *Strong (or Dry) Ten Martini Problem*. However, if non-collapsed gaps are dense in the spectrum, then this is still a Cantor set, although some (perhaps an infinite number) of collapsed gaps may also coexist.

Now we can formulate the second corollary in this paper:

Corollary 4 *Assume that $\omega \in \mathbb{R}$ is Diophantine. Then, there is a constant $C = C(\omega) > 0$ such that if $0 < |b| < C$ or $4/C < |b| < \infty$ all the spectral gaps of the spectrum of the Almost Mathieu operator are open.*

Before ending this introduction we give a short account of the existing results (to our knowledge) on the Cantor spectrum of the Almost Mathieu operator for $|b| \neq 0, 2$. The Cantor spectrum for the Almost Mathieu operator was first conjectured by Azbel [4] and Kac, in 1981, conjectured that all the spectral gaps are open. The problem of the Cantor structure of the spectrum was called the "Ten Martini Problem" by Simon [31] (and remained as Problem 4 in [32]). Sinai [33], proved that for Diophantine ω's and sufficiently large (or small $|b|$), depending on ω, the spectrum σ_b is a Cantor set. Choi, Elliott & Yui [8] proved that the spectrum σ_b is a Cantor set for all $b \neq 0$ when ω is a Liouville number obeying the condition

$$\left|\omega - \frac{p}{q}\right| < D^{-q},$$

for a certain constant $D > 1$ and infinitely many rationals p/q. In particular, this means that for a G_δ-dense subset of pairs (b, ω) the spectrum is a Cantor set, which is the Bellissard-Simon result [5]. For generic results on Cantor spectrum for almost periodic an quasi-periodic

Schrödinger operators see Moser [28] and Johnson [22]. Nevertheless, collapsed gaps appear naturally in quasi-periodic Schrödinger operators, as it was shown by Broer, Puig & Simó [6] and there are examples which do not display Cantor spectrum, see De Concini & Johnson [11]. Finally, let us mention that, if we consider the case of rational ω, all spectral gaps, apart from the middle one, are open if $b \neq 0$. This result was proved by van Mouche [34] and Choi, Elliott & Yui [8].

Let us now outline the contents of the present paper. In Section 2 we introduce some of the tools needed to prove our two main results. These include the different definitions of the rotation number, the concept of reducibility of linear quasi-periodic skew-products and the duality for the Almost Mathieu operator. In Section 3 apply the reducibility results by Eliasson to prove Corollary 4. Finally, in Section 4, the proof of of Corollary 2 is given, which is based on a result of non-perturbative localization by Jitomirskaya.

2 Prerequisites: rotation number, reducibility, duality and lack of coexistence

Rotation number

The rotation number for quasi-periodic Schrödinger equations is a very useful object with deep connections to the spectral properties of Schrödinger operators. It is also related to the dynamical properties of the solutions of the associated eigenvalue equation. This allows several equivalent definitions, which we shall now try to present.

The rotation number was introduced for continuous time quasi-periodic Schrödinger equations by Johnson & Moser [23]. The discrete version was introduced by Herman [18] (which is also defined for quasi-periodic skew-product flows on $SL(2,\mathbb{R}) \times \mathbb{T}$) and Delyon & Souillard [12] (which is the definition given in the introduction). We will now review these definitions, their connection and some important properties.

Herman's definition is dynamical. Here we follow the presentation by Krikorian [25]. Write equation (2) as a *quasi-periodic skew-product flow* on $\mathbb{R}^2 \times \mathbb{T}$,

$$u_{n+1} = A(\theta_n) u_n \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{3}$$

setting $u_n = (x_n, x_{n-1})^T$ and

$$A(\theta) = \begin{pmatrix} a - b\cos\theta & -1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

which belongs to $SL(2,\mathbb{R})$ the group of bidimensional matrices with determinant one. The quasi-periodic flow can also be defined on $SL(2,\mathbb{R}) \times \mathbb{T}$ considering the flow given by

$$X_{n+1} = A(\theta_n) X_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{5}$$

with $X_0 \in SL(2,\mathbb{R})$. This can be seen as the iteration of the following *quasi-periodic cocycle* on $SL(2,\mathbb{R}) \times \mathbb{T}$:

$$\begin{array}{rcl} SL(2,\mathbb{R}) \times \mathbb{T} & \longrightarrow & SL(2,\mathbb{R}) \times \mathbb{T} \\ (X,\theta) & \mapsto & (A(\theta)X, \theta + 2\pi\omega)\,, \end{array} \tag{6}$$

which we denote by (A, ω).

We will now give Herman's definition of the rotation number of a quasi-periodic cocyle like (6) with $A : \mathbb{T} \to SL(2,\mathbb{R})$ homotopic to the identity. For a general $A : \mathbb{T} \to SL(2,\mathbb{R})$, this last property is not always true, since $SL(2,\mathbb{R})$ is not simply connected. Indeed, its first homotopy group is isomorphic to $\mathbb{Z}$, with generator the rotation $R_1 : \mathbb{T} \to SL(2,\mathbb{R})$ given by

$$R_1(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for all $\theta \in \mathbb{T}$. In our case, the *Almost Mathieu cocyle* (4) is homotopic to the identity.

Let $\mathbb{S}^1$ be the set of unit vectors of $\mathbb{R}^2$ and let us denote by $p : \mathbb{R} \to \mathbb{S}^1$ the projection given by the exponential $p(t) = e^{it}$, identifying $\mathbb{R}^2$ with $\mathbb{C}$. Because of the linear character of the cocyle, the continuous map

$$\begin{array}{cccc} F: & \mathbb{S}^1 \times \mathbb{T} & \longrightarrow & \mathbb{S}^1 \times \mathbb{T} \\ & (v,\theta) & \mapsto & \left(\frac{A(\theta)v}{\|A(\theta)v\|}, \theta + 2\pi\omega\right) \end{array} \tag{7}$$

is also homotopic to the identity. Therefore, it admits a continuous lift $\tilde{F} : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ of the form:

$$\tilde{F}(t,\theta) = (t + f(\theta,t), \theta + 2\pi\omega)$$

such that

$$f(t + 2\pi, \theta + 2\pi\omega) = f(t,\theta) \text{ and } p\,(t + f(t,\theta)) = \frac{A(\theta)p(t)}{\|A(\theta)p(t)\|}$$

for all $t \in \mathbb{R}$ and $\theta \in \mathbb{T}$. The map f is independent of the choice of $\tilde{F}$ up to the addition of a constant $2\pi k$, with $k \in \mathbb{Z}$. Since the map $\theta \mapsto \theta + 2\pi\omega$ is uniquely ergodic on $\mathbb{T}$ for all $(t,\theta) \in \mathbb{R} \times \mathbb{T}$, the limit

$$\lim_{N\to\infty} \frac{1}{2\pi N} \sum_{n=0}^{N-1} f\left(\tilde{F}^n(t,\theta)\right)$$

exists, it is independent of (t,θ) and the convergence is uniform in (t,θ), see Herman [18] and Johnson & Moser [23]. This object is called the *fibered rotation number*, which will be denoted as $\rho_f(a,b)$, and it is defined modulus $\mathbb{Z}$.

For instance, if $A_0 \in SL(2,\mathbb{R})$ is a constant matrix, then the rotation number of the cocycle (A_0,ω), for any irrational ω, is the absolute value of the argument of the eigenvalues divided by 2π.

Using a suspension argument (see Johnson [24]) it can be seen that, for the Almost Mathieu cocycle (like for any quasi-periodic Schrödinger cocycle), the fibered rotation number coincides with the *Sturmian* definition given in the introduction. Note that this last rotation number, $\mathrm{rot}(a,b)$, belongs to the interval, whereas the fibered rotation number, $\varrho_f(a,b)$, is an element of $\mathbb{R}/\mathbb{Z}$. They can be both linked by means of the *integrated density of states*, see Avron & Simon [3]. Let $k_L(a,b,\phi)$ be $(L-1)^{-1}$ times the number of eigenvalues less than or equal to a for the restriction of $H_{b,\phi}$ to the set $\{1,\ldots,L-1\}$, for some $\phi \in \mathbb{T}$, with zero boundary conditions at both ends 0 and L. Then, as $L \to \infty$, the $k_L(a,b,\phi)$ converge to a continuous function $k(a,b)$, which is the integrated density of states. The basic relations are

$$2\mathrm{rot}(a,b) = k(a,b) \qquad \text{and} \qquad 2\varrho_f(a,b) = k(a,b) + l,$$

for a suitable integer $l \in \mathbb{Z}$. In particular,

$$\mathrm{rot}(a,b) = \rho_f(a,b) \pmod{\tfrac{1}{2}\mathbb{Z}}.$$

In what follows, the arithmetic nature of the rotation number will be of importance. We will say that the rotation number is *rational or resonant* with respect to ω if there exists a constant $k \in \mathbb{Z}$ such that $\mathrm{rot}(a,b) = \{k\omega\}/2$ or equivalently, $\varrho_f(a,b) = k\omega/2$ modulus $\frac{1}{2}\mathbb{Z}$. Also, we say that it is *Diophantine* with respect to ω whenever the bound

$$\left|\mathrm{rot}(a,b) - \frac{\{k\omega\}}{2}\right| = \min_{l\in\mathbb{Z}}\left|\rho_f(a,b) - \frac{k\omega}{2} - \frac{l}{2}\right| \geq \frac{K}{|k|^\tau},$$

holds for all $k \in \mathbb{Z} - \{0\}$ and suitable fixed positive constants K and τ.

Reducibility

A main tool in the study of quasi-periodic skew-product flows is its *reducibility* to constant coefficients. Reducibility is a concept defined for the continuous and discrete case (for an introduction see the reviews by Eliasson [14], [15] and, for more references, the survey [30] by the author).

A quasi-periodic skew-product flow like (3), or a quasi-periodic cocycle like (6), with $A : \mathbb{T} \to SL(2,\mathbb{R})$, is said to be *reducible to constant coefficients* if there is a continuous map $Z : \mathbb{T} \to SL(2,\mathbb{R})$ and a constant matrix $B \in SL(2,\mathbb{R})$, called the Floquet matrix, such that the conjugation

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B \tag{8}$$

is satisfied for all $\theta \in \mathbb{T}$. When ω is rational, in which case the flow is periodic, any skew-product flow is reducible to constant coefficients. Even in this periodic case, it is not always possible to reduce with the same frequency ω, but with $\omega/2$. If there is a reduction to constant coefficients like (8), then a fundamental matrix of solutions of (3),

$$X_{n+1}(\phi) = A(2\pi n\omega + \phi)X_n(\phi), \qquad n \in \mathbb{Z},$$

with $X_0 : \mathbb{T} \to SL(2,\mathbb{R})$ continuous, has the following *Floquet representation*:

$$X_n(\phi) = Z(2\pi n\omega + \phi)B^n Z(\phi)^{-1}X_0(\phi) \tag{9}$$

for all $n \in \mathbb{Z}$ and $\phi \in \mathbb{T}$. This gives a complete description of the qualitative behaviour of the flow (3).

The rotation number of a quasi-periodic cocycle is not invariant through a conjugation like (8). There are however the following easy relations:

Proposition 5 *Let ω be an irrational number and (A_1,ω) and (A_2,ω) be two quasi-periodic cocycles on $SL(2,\mathbb{R})\times$ homotopic to the identity, being ρ_1 and ρ_2 the corresponding fibered rotation numbers. Assume that there exists a continuous map $Z : \mathbb{T} \to SL(2,\mathbb{R})$ such that*

$$A_1(\theta)Z(\theta) = Z(\theta + 2\pi\omega)A_2(\theta)$$

for all $\theta \in \mathbb{T}$. Then, if $k \in \mathbb{Z}$ is the degree of Z,

$$\rho_1 = \rho_2 + k\alpha \text{ modulus } \mathbb{Z}.$$

This proposition shows that, for any fixed irrational frequency ω, the class of quasi-periodic cocycles with rational rotation number (resp. with Diophantine rotation number) is invariant under conjugation, although the rotation number itself may change. Also, that whenever a quasi-periodic skew-product flow in $SL(2,\mathbb{R}) \times \mathbb{T}$ is reducible to a Floquet matrix with trace ± 2, the rotation number must be rational.

Duality and lack of coexistence

To end this section, let us present a specific feature of the Almost Mathieu operator or, rather, of the associated eigenvalue equation which will in the basis of our arguments. It is part of what is known as *Aubry duality* or simply *duality*:

Theorem 6 (Avron & Simon [3]) *For every irrational ω, the rotation number of (2) satisfies the relation*

$$\mathrm{rot}(a, b) = \mathrm{rot}(2a/b, 4/b) \tag{10}$$

for all $b \neq 0$ and $a \in \mathbb{R}$.

According to Proposition 3 this means that the spectrum $\sigma_{4/b}$, for $b \neq 0$ is just a dilatation of the spectrum σ_b. In particular, σ_b is a Cantor set (resp. none of the spectral gaps of σ_b is collapsed) if, and only if $\sigma_{4/b}$ is a Cantor set (resp. none of the spectral gaps of $\sigma_{4/b}$ is collapsed).

In the proof of our two main results we will use the following argument, which is analogous to Ince's argument for the classical Mathieu periodic differential equation (see [19] §7.41). In principle, the eigenvalue equation of a general quasi-periodic Schrödinger operator may have two linearly independent quasi-periodic solutions with frequency ω (or $\omega/2$). One may call this phenomenon *coexistence* of quasi-periodic solutions, in analogy with the classical Floquet theory for second-order periodic differential equations. A trivial example of this occurs in the Almost Mathieu case for $b = 0$ and suitable values of a.

Let us now show that in the Almost Mathieu case this does not happen if $b \neq 0$, i.e. two quasi-periodic solutions with frequency ω of the eigenvalue equation cannot coexist. Let $(x_n)_{n\in\mathbb{Z}}$ satisfy the equation

$$x_{n+1} + x_{n-1} + b\cos(2\pi\omega n + \phi)x_n = ax_n, \qquad n \in \mathbb{Z} \tag{11}$$

for some a, $b \neq 0$ and ϕ. If it is quasi-periodic with frequency ω, there exists a continuous function $\psi : \mathbb{T} \to \mathbb{R}$ such that $x_n = \psi(2\pi\omega n + \phi)$ for all $n \in \mathbb{Z}$. The Fourier coefficients of ψ, $(\psi_m)_{m\in\mathbb{Z}}$ satisfy the following equation:

$$2\cos(2\pi\omega m)\psi_m + \frac{b}{2}(\psi_{m+1} + \psi_{m-1}) = a\psi_m, \qquad m \in \mathbb{Z},$$

which is equivalent to

$$\psi_{m+1} + \psi_{m-1} + \frac{4}{b}\cos(2\pi\omega m)\psi_m = \frac{2a}{b}\psi_m, \qquad m \in \mathbb{Z}. \tag{12}$$

Since ψ is at least continuous, then $(\psi_m)_{m\in\mathbb{Z}}$ belongs to $l^2(\mathbb{Z})$. Now the reason for the absence of coexisting quasi-periodic solutions is clear. Indeed, if if $(y_n)_{n\in\mathbb{Z}}$ is another linearly independent quasi-periodic solution of (11) with frequency ω, say $y_n = \chi(2\pi\omega + \phi)$, for some continuous χ, then the sequence of the Fourier coefficients of χ, $(\chi_m)_{m\in\mathbb{Z}}$, would be a solution of (12) belonging to $l^2(\mathbb{Z})$. The sequences $(\psi_m)_{m\in\mathbb{Z}}$ and $(\chi_m)_{m\in\mathbb{Z}}$ would be two linearly independent solutions of (12) which belong both to $l^2(\mathbb{Z})$.

This is a contradiction, because for bounded potentials, like the cosine, we are always in the limit-point case (see [7, 9] for the continuous case). In our discrete case, this is even simpler, since any solution in $l^2(\mathbb{Z})$ of the eigenvalue equation must tend to zero at $\pm\infty$. Hence, the existence of two linearly independent solutions belonging both to $l^2(\mathbb{Z})$ would be in contradiction with the preservation of the Wronskian.

Therefore, two quasi-periodic solutions with frequency ω cannot coexist if $b \neq 0$. A similar argument shows that quasi-periodic solutions of the form

$$(-1)^n \psi(2\pi\omega n + \phi), \tag{13}$$

for a continuous $\psi : \mathbb{T} \to \mathbb{R}$ cannot coexist.

Finally, note that the coexistence of two quasi-periodic solutions with frequency ω of equation (11) is equivalent to the reducibility of the corresponding two-dimensional skew-product flow (3), with the identity as Floquet matrix. Similarly the coexistence of two quasi-periodic solutions of the type (13) is equivalent to the reducibility of the flow with minus the identity as Floquet matrix.

3 The Strong Ten Martini Problem for small (and large) $|b|$

In this section we will show that for $0 < |b| < C$, where $C > 0$ is a suitable constant, and for $|b| > 4/C$ all spectral gaps are open.

The theorem from which we will derive Corollary 4 is due to Eliasson and it was originally stated for the continuous case, based on a KAM scheme. It can be adapted to the discrete case to obtain the following:

Theorem 7 ([16, 17]) *Assume that ω is Diophantine with constants c and r. Then there is a constant $C(c,r)$ such that, if $|b| < C(c,r)$ and $\mathrm{rot}(a,b)$ is either rational or Diophantine, then the quasi-periodic skew-product flow*

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} a - b\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{14}$$

on $\mathbb{R}^2 \times \mathbb{T}$ is reducible to constant coefficients, with Floquet matrix B, by means of a quasi-periodic (with frequency $\omega/2$) and analytic transformation. Moreover, if a is at an endpoint of a spectral gap of σ_b, then the trace of B is ± 2, being $B = \pm I$ if, and only if, the gap collapses. Finally, if $B = \pm I$ then the transformation Z can be chosen to have frequency ω.

For other reducibility results in the context of quasi-periodic Schrödinger operators see Dinaburg & Sinai [13] and Moser & Pöschel [29] for the continuous case and Krikorian [25] and Avila & Krikorian [2] for the discrete case.

Taking into account the arguments from the previous section, Corollary 4 is immediate. Indeed, let $|b| < C$, where C is the constant given by the theorem for a fixed Diophantine frequency ω. Then the skew-product flow (14) is reducible to constant coefficients and the Floquet matrix has trace ± 2 if a is an endpoint of a spectral gap. Moreover the gap is collapsed if, and only if, the Floquet matrix B is $\pm I$. Since we have seen in the previous section that (14) for $b \neq 0$ cannot be reducible to these Floquet matrices, Corollary 4 follows.

4 Non-perturbative localization and Cantor spectrum for $b \neq 0$

In this section we will see how Corollary 2 is a consequence of the following theorem on non-perturbative localization, due to Jitomirskaya:

Theorem 8 ([20]) *Let ω be Diophantine. Define the set Φ of* resonant phases *as the set of those $\phi \in \mathbb{T}$ such that the relation*

$$|\sin(\phi + \pi n\omega)| < \exp\left(-|n|^{\frac{1}{2r}}\right) \tag{15}$$

holds for infinitely many values of n, being r the constant in the definition of a Diophantine number. Then, if $\phi \notin \Phi$ and $|b| > 2$ the operator $H_{b,\phi}$ has only pure point spectrum with exponentially decaying eigenfunctions. Moreover, any of these eigenfunctions $(\psi_n)_{n\in\mathbb{Z}}$ satisfies that

$$\beta(b) = -\lim_{|n|\to\infty} \frac{\log\left(\psi_n^2 + \psi_{n+1}^2\right)}{2|n|} = \log\left(\frac{|b|}{2}\right). \tag{16}$$

Now we prove Corollary 2. Let $|b| > 2$. Then, according to Theorem 8, the operators $H_{b,0}$ and $H_{b,\pi/2}$ have only pure point spectrum with exponentially decaying eigenfunctions. The eigenvalue equation associated to these operators has the following properties:

Lemma 9 *Let $(x_n)_{n\in\mathbb{Z}}$ be a solution of the difference equation*

$$x_{n+1} + x_{n-1} + b\cos(2\pi n\omega + \phi)x_n = ax_n, \qquad n \in \mathbb{Z},$$

for some constants a, b and $\phi \in \mathbb{T}$. Then, if $\phi = 0$, $(x_{-n})_{n\in\mathbb{Z}}$ is also a solution of this equation and, if $\phi = -\pi/2$ then $(-x_{-n})_{n\in\mathbb{Z}}$ is also a solution.

Let us consider first the case of the operator $H_{b,0}$. According to Theorem 8, there exists a sequence of eigenvalues $(a^k(b))_{k\in\mathbb{Z}}$ with eigenvectors $(\psi^k(b))_{k\in\mathbb{Z}}$, exponentially localized and which form a complete orthonormal basis of $l^2(\mathbb{Z})$. Moreover the set of eigenvalues $(a^k(b))_{k\in\mathbb{Z}}$ must be dense in the spectrum σ_b. Again, we do not write the dependence on b for simplicity in what follows. None of these eigenvalues can be repeated, since we are in the limit point case. Writing each of the ψ^k as

$$\psi^k = (\psi_n^k)_{n\in\mathbb{Z}},$$

we define

$$\tilde{\psi}^k(\theta) = \sum_{k\in\mathbb{Z}} \psi_n^k e^{ik\theta},$$

for $\theta \in \mathbb{T}$. All these functions belong to $C_\beta^a(\mathbb{T}, \mathbb{R})$, the set of real analytic functions of $\mathbb{T}$ with analytic extension to $|\Im\theta| < \beta$ and they are even functions of θ, because of Lemma 9 (here we have applied again that we are in the limit point case). Passing to the dual equation, we obtain that, for each $k \in \mathbb{Z}$, the sequence $(\tilde{\psi}^k(2\pi\omega n))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of

$$x_{n+1} + x_{n-1} + \frac{4}{b}\cos\theta_n x_n = \frac{2a}{b}x_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \qquad n \in \mathbb{Z}, \tag{17}$$

provided a is now replaced by a^k. We are now going to see that $2a^k/b$ is at an endpoint of a spectral gap and that this is collapsed. To do so we will use reducibility as in the proof of Theorem 4. For a direct proof that $2a^k/b$ is at an endpoint of a gap (it has rational rotation number), see again Herman [18].

The fact that $(\tilde{\psi}^k(2\pi\omega n))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of (17) means that, for all $\theta \in \mathbb{T}^d$, the following equation is satisfied

$$\begin{pmatrix} \tilde{\psi}^k(4\pi\omega + \theta) \\ \tilde{\psi}^k(2\pi\omega + \theta) \end{pmatrix} = \begin{pmatrix} \frac{2a}{b} - \frac{4}{b}\cos\theta & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}^k(2\pi\omega + \theta) \\ \tilde{\psi}^k(\theta) \end{pmatrix}.$$

The following lemma shows that, if this is the case, then the quasi-periodic skew-product flow

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a^k}{b} - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{18}$$

is reducible to constant coefficients.

Lemma 10 *Let $A : \mathbb{T} \to SL(2,\mathbb{R})$ be a real analytic map, with analytic extension to $|\Im\theta| < \delta$ for some $\delta > 0$. Assume that there is a nonzero real analytic map $v : \mathbb{T} \to \mathbb{R}^2$, with analytic extension to $|\Im\theta| < \delta$ such that*

$$v(\theta + 2\pi\omega) = A(\theta)v(\theta)$$

holds for all $\theta \in \mathbb{T}$. Then, the quasi-periodic skew-product flow given by

$$u_{n+1} = A(\theta_n)u_n, \qquad \theta_{n+1} = \theta_n + 2\pi\omega, \tag{19}$$

with $(u_n, \theta_n) \in \mathbb{R}^2 \times \mathbb{T}$ for all $n \in \mathbb{Z}$ is reducible to constant coefficients by means of a quasi-periodic transformation which is analytic in $|\Im\theta| < \delta$ and has frequency ω. Moreover the Floquet matrix can be chosen to be of the form

$$B = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{20}$$

for some $c \in \mathbb{R}$.

Proof: Since $v = (v_1, v_2)^T$ does not vanish, $d = v_1^2 + v_2^2$ is always different from zero and the transformation

$$Z(\theta) = \begin{pmatrix} v_1(\theta) & -v_2(\theta)/d(\theta) \\ v_2(\theta) & v_1(\theta)/d(\theta) \end{pmatrix},$$

is an analytic map $Z : \mathbb{T} \to SL(2,\mathbb{R})$. The transformation Z defines a conjugation of A with B^1, being

$$A(\theta)Z(\theta) = Z(2\pi\omega + \theta)B^1(\theta),$$

which means that B^1 is

$$B^1(\theta) = \begin{pmatrix} 1 & b_{12}^1(\theta) \\ 0 & 1 \end{pmatrix},$$

for some analytic $b_{12}^1 : \mathbb{T} \to \mathbb{R}$. The transformed skew-product flow, defined by the matrix B^1 is reducible to constant coefficients because it is in triangular form, the frequency ω is Diophantine and b_{12}^1 is analytic. Indeed, if $y_{12} : \mathbb{T} \to \mathbb{R}$ is an analytic solution of the small divisors equation

$$y_{12}(2\pi\omega + \theta) - y_{12}(\theta) = b_{12}^1(\theta) - [b_{12}^1], \qquad \theta \in \mathbb{T},$$

where $[b_{12}^1]$ is the average of b_{12}^1 (see [1]), then the transformation

$$Y(\theta) = \begin{pmatrix} 1 & y_{12} \\ 0 & 1 \end{pmatrix}$$

conjugates B^1 with the its averaged part:

$$B = [B^1] = \begin{pmatrix} 1 & [b_{12}^1] \\ 0 & 1 \end{pmatrix}$$

which is in the form of (20).□

Thus, applying this lemma, the flow (18) is reducible to constant coefficients with Floquet matrix B, of the form (20). That is, there exists a real analytic map $Z : \mathbb{T} \to SL(2,\mathbb{R})$ such that

$$A(\theta)Z(\theta) = Z(\theta + 2\pi\omega)B \tag{21}$$

for all $\theta \in \mathbb{T}$. Moreover, since the trace of B is 2, the rotation number of (17) is rational, so that we are at the endpoint of a gap, which we want to show that is non-collapsed.

By the arguments of Section 2, we rule out the possibility of B being the identity. Indeed, this would imply the coexistence of two quasi-periodic analytic solutions with frequency ω, which does happen in the Almost Mathieu case. Therefore $B \neq I$ and, thus, $c \neq 0$ in the definition above.

If $B \neq I$, it is a well-known fact of Floquet theory that $2a^k/b$ lies at the endpoint of a non-collapsed gap (see, for example, the monograph [35] for classical Floquet theory or [6] for the continuous and quasi-periodic Schrödinger case). For the sake of self-completeness we sketch the argument.

We will see that there exists a $\alpha_0 > 0$ such that if $0 < |\alpha| < \alpha_0$ and α is either positive or negative (depending on the sign of c) then $2a^k/b + \alpha$ lies in the resolvent set of $\sigma_{4/b}$. To do so, we will show that, for these values of α, the skew-product flow

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{2a^k}{b} + \alpha - \frac{4}{b}\cos\theta_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \qquad \theta_{n+1} = \theta_n + 2\pi\omega \tag{22}$$

has an exponential dichotomy (see Coppel [10]) which implies that $2a^k/b + \alpha \notin \sigma_{4/b}$ (see Johnson [21]). The reduction given by Z transforms this system into

$$\begin{aligned} y_{n+1} &= \begin{pmatrix} 1 + \alpha\,(z_{11}z_{12} - cz_{11}^2) & c + \alpha\,(-cz_{11}z_{12} + z_{12}^2) \\ -\alpha z_{11}^2 & 1 - \alpha z_{11}z_{12} \end{pmatrix} y_n, \\ \theta_{n+1} &= \theta_n + 2\pi\omega, \end{aligned} \tag{23}$$

where $y_n \in \mathbb{R}^2$ are the new variables. The z_{ij} are the elements of the matrix Z and we have used the relations given by (21) and the special form of A and B. In the same calculation, we also see that $(z_{11}(2\pi n\omega))_{n\in\mathbb{Z}}$ is a quasi-periodic solution of equation (17) and that it is not identically zero. Using averaging theory (see, for example, [1]), system (23) can be transformed into

$$\begin{aligned} y_{n+1} &= \left(\begin{pmatrix} 1 + \alpha\,([z_{11}z_{12}] - c[z_{11}^2]) & c + \alpha\,(-c[z_{11}z_{12}] + [z_{12}^2]) \\ -\alpha[z_{11}^2] & 1 - \alpha[z_{11}z_{12}] \end{pmatrix} + M \right) y_n \\ \theta_{n+1} &= \theta_n + 2\pi\omega \end{aligned} \tag{24}$$

by means of a conjugation in $SL(2,\mathbb{R})$, with M analytic in both θ and α (in some narrower domains) and of order α^2. The time-independent part of the above system is hyperbolic if $c\alpha < 0$. Therefore, if $|\alpha| \neq 0$ is small enough the time-dependent system (24) has an exponential dichotomy for $c\alpha < 0$. Hence $2a^k/b + \alpha$ does not belong to $\sigma_{4/b}$. Since this works for all a^k, (which are dense in the spectrum), $\sigma_{4/b}$ is a Cantor set. By duality the result is also true for σ_b. This ends the proof of Corollary 2.

Remark 11 *The same can be done for the operator $H_{b,\pi/2}$ instead of $H_{b,0}$. In this case the Floquet matrix has trace -2. The corresponding point eigenvalues correspond to ends of non-collapsed gaps and are dense in the spectrum.*

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